Motivation

The idea of constructing random probability measures by normalizing the increments of a random process dates back to Kingman’s construction of the Dirichlet process by normalization of the increments of a Gamma process (Kingman, 1975). Recently the normalization approach has gained new interest in Bayesian nonparametrics, when a key problem is to define tractable alternatives to the Dirichlet process to be used as a prior on the space of probability distributions (see e.g. James et al. 2005). A comprehensive Bayesian analysis of a very large class of random probability measures obtained by an appropriate normalization is proposed in James et al. (2005) by providing explicit tractable alternatives to the Blackwell-Matomäki-Fiuza semi parametric random measures and applicable descriptives of posterior distributions. In this context the Gibbs product form of the EPP (and consequently of the predictive distributions), is highly desirable both with regard to mathematical tractability of posterior distributions, like in connection with sequential sampling schemes for posterior’s simulation, hence it is interesting to characterize the class of normalized random measures which possess this property.

Main Result

Relying on definitions and constructions recalled in the two side columns, the following proposition can be stated:

Proposition 1. Let \( \mu \) be a density on \((0, \infty)\), corresponding to a strictly positive infinitely divisible rv \( T \), \( T \neq \gamma \) for a probability density on \((0, \infty)\) and \( h \) a fixed and non-atomic distribution on \((S, \mathcal{A})\). Then a mixed \( PK \) model on \( S \times (0, \infty) \) can be constructed by normalizing \( P \) as follows

\[
P(\cdot, \mu) = \frac{1}{\mathbb{E}[T]} \int_{0}^{\infty} \mathbb{P}(x; \mu) \mathbb{P}(dx)
\]

As proved e.g. in James (2003) NRM select almost surely discrete distributions, and it is well known that given a law \( Q \) on the Polish space \( S \), a random discrete probability measure \( (RDPM) \) \( P \) on \( S \) may always be defined as

\[
P(\cdot, \mu) = \sum_{j=1}^{\infty} \mathbb{P}(\cdot, \mu_j)
\]

Pitman termed the law \( Q \) on \( S \) of a Poisson-Kingman distribution with density \( \mu \) and mixing distribution \( \gamma \) by \( \text{PK}(\mu, \gamma) \). For \( \alpha > 1 \), \( \mu_j \) are atoms of size \( (\alpha - 1) / \alpha \) and \( (\alpha - 1) / \alpha \) is the mixing density. For \( \alpha = 1 \), \( \mu \) is a uniform distribution on \((0, 1)\).

2. Exchangeable Gibbs Partitions. From Kingman’s (1978) theory of exchangeable random partitions, sampling from a RDPM \( P \) induces a random partition \( \Pi \) of the positive integers \( \mathbb{N} \) by the exchangeable equivalence relation \( \sim \). The distribution \( (RDPM) \) of the partition \( \Pi \) is determined by \( P \). Now, by construction, \( \Pi \) is exchangeable.

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Hence exchangeable sequences driven by PK models are completely identified by the corresponding EPP, a particularly tractable case (2) arises when the EPP of Gibbs form, (cfr. Def. 1 in Gnedin and Pitman, 2006), i.e. for all \( 1 \leq i \leq n \), and all compositions \( (n_1, \ldots, n_k) \) of \( n \), Gnedin and Pitman (2006) also show that to define an infinite random partition of \( \mathbb{N} \), i.e. a sequence \((\ell_n)\) consistent as \( n \) varies, the weights \( \left\{ \mathbb{W}_l \right\} \) must be of the following very simple form depending on a single parameter \( \alpha \in (0, 1) \), \( \mathbb{W}_k = (\alpha - 1) / \alpha \), and \( \mathbb{W}_0 \) must be the solution to the backward recursion \( \mathbb{W}_j = (\alpha - 1) / \alpha \),\( (\Pi_j = (\alpha - 1) / \alpha) \) with \( \mathbb{W}_j = 1 \).

For \( (\alpha, \beta) \in (0, 1) \), \( \alpha < \beta \), \( E_{\mathbb{W}, \mathbb{K}} \) is called the \( \alpha \)-stable subordinator

\[
\mathbb{P}(T \geq t) = 1 - \frac{\mathbb{E}[T]}{t^{\beta - 1}}
\]

with corresponding Laplace exponent \( \phi(x) = x^{\beta - 1} \phi_{\mathbb{E}}(x) \).

An explicit expression for the sequence of predictive distributions can be easily deduced (specializing (4) for \( \alpha = 1 / 2 \) and \( \beta = 1 / 2 \).)

\[
q_{\alpha, \beta}(\delta) = \frac{1}{\sqrt{\pi}} \left( \frac{\delta}{\alpha} \right)^{\alpha - 1 / 2} \exp \left( - \frac{\delta^{2 \beta}}{\alpha} \right)
\]

and, by Proposition 2, \( PK(\mathbb{E}, \mathbb{K}) \) is

\[
\mathbb{P}(x | \Pi) = \frac{1}{\mathbb{E}[T]} \sum_{i=0}^{\infty} \mathbb{P}(x | \Pi_i) \mathbb{P}(dx)
\]

with some manipulations, and having at hand the definition of the complete Gamma function, i.e. \( \Gamma(x) = \int_{0}^{\infty} t^{x - 1} \exp(-t) dt \), it is easy to see that (9) reduces to formula (A1) in Appendix A.4 of Lijoi et al. (2005), and results in Proposition 3 arise by specializing (8) for \( \alpha = 1 / 2 \).

References


